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ECONOMIC DEVELOPMENT AND SUSTAINABILITY IN A TWO-SECTOR MODEL WITH VARIABLE POPULATION GROWTH RATE

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Abstract

In this paper, we study the neoclassical Solow-Swan model where the natural capital is introduced as a factor of production and modeled as a renewable resource. In contrast with the standard literature, the labor growth rate is assumed to be non constant over time. In this framework, we investigate the conditions under which the economy may be sustainable or unsustainable in the long run, we derive the set of sustainable marginal propensity to consume for any given tax rate, we determine the nature of the non-trivial steady states of the economy.

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1. Introduction

Long-run growth was first introduced by Solow [7] and Swan [8] into the traditional neoclassical macroeconomic model by considering a growing population coupled with a more efficient labor force. Their articles presented a mathematical model, in the form of a differential equation, describing how increased capital stock generates greater per capita production. From simply being a tool for the analysis of the growth process, the Solow-Swan model has been generalized in several different directions (see, for example, Hall and Taylor [4]; Mankiw [5]; Romer [6]), but, as noticed by Dasgupta [2], with no mention of environmental resources. An effort to address this omission was done by Tran-Nam [9], who considered an infinite-horizon aggregative closed economy where the production function depends on physical capital, natural capital and labor, and showed that if human activities have a net zero or negative effect on the environment, then the economy is unsustainable in the long-run, in the sense that physical and natural capital per worker will tend to zero as time grows indefinitely large. Furthermore, if human activities produce a net beneficial effect on the environment, then the economy will converge to a unique and stable steady state. A natural question to be asked in Tran-Nam's model is what the impact of changes in the population growth rate would be, i.e., to examine the consequences of relaxing the assumption of constant population growth rate. This assumption is not a good approximation to reality. The main problem is that population grows exponentially, and so tends to infinity as time goes to infinity, which is clearly unrealistic. Following Guerrini [3], we consider a more realistic approach by assuming the labor growth rate to be variable over time and controllable subject to be between prescribed upper and lower limits. The natural capital stock is modeled as a renewable resource, so that we make no distinction between resource and environmental economists. Resource economists, who are interested in population ecology, characterize complex systems by the population sizes of different species, while environmental economists, who are interested in systems ecology, summarize complex systems in terms of indices of quality of air, soil or water. Here, we combine both approaches by treating the environmental capital as a stock of measurable in some constant quality units. Since we focus on economic theory, all practical problems associated with measuring natural capital are assumed away. The change in the stock of natural capital depends on its autonomous evolution, production and consumption externalities, and environment maintenance programs. By modeling the natural capital stock as a renewable resource, we have that damages done to the environment production and consumption externalities are reversible, and can be corrected by collective maintenance actions. In this framework, we find out that the long run sustainability of the economy depends crucially on human activities and on the stock of the environment. We have that the economy is sustainable in the long run if human activities have a net zero effect on the environment and the stock of the environment grows or remains unchanged autonomously over time, it is unsustainable if human activities have a net zero or negative effect on the environment and the stock of the environment decays autonomously over time. For any given tax rate (or marginal propensity to consume, say MPC), we derive the set of sustainable MPCs (or tax rates). Finally, we examine the non-trivial steady states of a sustainable economy, and discover that there are infinite unstable steady states equilibrium if human activities have a net zero effect on the environment and the environment remains unchanged over time, while there is a unique stable steady state equilibrium, which is a saddle, or a node, if human activities have a negative effect on the environment and the environment grows over time, or if human activities have a positive effect on the environment and the environment decays over time, respectively.

2. The Model

We consider a closed economy in continuous time where a homogeneous goods is produced according to a technology which involves three inputs: physical capital, natural capital and labor, and satisfies all neoclassical properties, i.e., it has positive and diminishing marginal products with respect to each input, it exhibits constant returns to scale, and it satisfies the Inada conditions. The production function is specified as

$$Y(t) = K(t)^{\eta} E(t)^{\theta} L(t)^{1-\eta-\theta}, \ \eta, \ \theta, \ \eta + \theta \in (0, 1),$$

where $Y(t)$ is the flow of output, $K(t)$ is the stock of physical capital, $E(t)$ is the stock of natural capital, and $L(t)$ is the labor force. In all subsequent equations, the time argument is suppressed to ease the burden of notations. Based on the feature of constant returns to scale, Y may be written in terms of capital per worker as $y = k^{\eta} e^{\theta}$, where $y = Y/L$ is the per capita capital output, $k = K/L$ denotes the physical capital stock per worker, and $e = E/L$ is the natural capital stock per worker. Output is assumed to be used for consumption C, for savings S, or spent to maintain or improve the environment. The national accounting is given by

$$Y = C + S + T,$$

where S and T stand for saving and tax, respectively. We assume that the tax revenue is a constant fraction of output, i.e., $T = \tau Y$, with $\tau \in (0, 1)$ the tax rate, and the consumption is a constant fraction of disposable income, i.e., $C = a(Y - T)$, with $a \in (0, 1)$ the marginal propensity to consume (MPC). This yields $C = a(1 - \tau)Y$. Next, a constant fraction δ of the capital stock depreciates every period, meaning that if at the beginning of a period the capital stock equals K at the end of it, δK will have been worn off. Thus, the net increase in capital stock at any moment in time is equal to the amount of gross investment I less the amount of depreciated capital, i.e., $\dot{K} = I - \delta K$, where a dot over a variable denotes time derivative. Since the economy is closed, the output of the economy equals total income, whereas investments equal savings. Consequently, the capital accumulation equation takes the form

$$\dot{K} = (1 - a)(1 - \tau)Y - \delta K. \tag{1}$$

By differentiating K/L with respect to time, we get

$$\dot{k} = \frac{d(K/L)}{dt} = \frac{\dot{K}}{L} - k\frac{\dot{L}}{L}. \tag{2}$$

Substituting equation (1) in equation (2) yields that the instantaneous change in the physical capital stock per capita and in the natural capital stock at any moment is given by

$$\dot{k} = (1 - \alpha)(1 - \tau)k^{\eta}e^{\theta} - \left(\delta + \frac{\dot{L}}{L}\right)k.$$

In general, population is assumed to equal the labor force L, and growth according to $\dot{L} = nL$, where n is the given population growth rate. The main problem of this assumption is that population grows exponentially, $L(t) = L(0)e^{nt}$, and so tends to infinity as time goes to infinity, which is clearly unrealistic. Contrary to the standard literature, we consider a more realistic approach by assuming that $\dot{L}$ at any moment of time is a function of the population size L at that moment, i.e., $\dot{L} = g(L)$. Since a zero population has a zero growth, $L = 0$ is an algebraic root of the function $g(L)$. Thus, we may write

$$\dot{L} = Ln(L),$$

where $n(L)$ is a function of L. Following Guerrini [3], we assume $n(L)$ to be controllable subject to be between prescribed upper and lower limits, i.e.,

$$0 \leq n(L) \leq M, \lim_{t \to \infty} n(L) = 0, n_L(L) < 0.$$

In particular, we have that $L(0) \leq L(t) \leq L(0)e^{Mt}$, for all t. Furthermore, let us assume $L(0) = 1$, $\lim_{t \to \infty} L = L_{\infty} < \infty$, and there exists a unique value $L_* \neq 0$ such that $n(L) = 0$. An example of such a population growth rate is provided by the well-known logistic map (see Verhulst [10]).

Remark 1. $L_{\infty} < \infty$, yields $n(L_{\infty}) = 0$. This statement from the next result:

"Let $\varphi : [x_0, +\infty) \to \mathbb{R}$ be a differentiable function such that there exist (finite or infinite) the limits $\lim_{x \to +\infty} \varphi(x) = l$, $\lim_{x \to +\infty} \varphi'(x) = n$. If l is finite, then $n = 0$."

The proof is as follows. By Lagrange's theorem, $\varphi(x+1)-\varphi(x)=\varphi'(\xi_x)$, for some $\xi_x \in (x, x+1)$. Since $\lim_{x\to+\infty} \xi_x = +\infty$, we have that $\lim_{x\to+\infty} \varphi'(\xi_x) = \lim_{x\to+\infty} \varphi'(x) = n$. Thus, $\lim_{x\to+\infty} [\varphi(x+1)-\varphi(x)] = n$. Next, l finite implies $\lim_{x\to+\infty} [\varphi(x+1)-\varphi(x)] = 0$.

Note that $n(L_\infty) = 0$, and so $L_\infty = L_*$. Regarding the environmental stock E, we assume that its evolution over time is described by the following differential equation

$$\dot{E} = \alpha E + \phi T - \beta Y - \gamma C, \tag{3}$$

where $\alpha, \phi, \beta,$ and γ are some constants. In case there is no human economic activity, E changes over time at the exponential rate α, with the parameter α positive, zero or negative according to whether the environment grows, remains unchanged or decays autonomously over time. In case there is human economic activity, we have depletion of β units of E for every unit of the final goods produced (the production of the final goods causes external damages to the environment), and also that each unit of the final goods consumed depletes γ units of the environmental stock. Furthermore, environmental programs, funded by the entire tax revenue, generate ϕ units of the environmental stock per unit of the tax spent. The government runs a balanced budget at any instant of time, the taxation revenue is costlessly collected, and there are no government failures. In order to express the equation (3) in per capita terms, first we rewrite it as

$$\dot{E} = \alpha E + [(\phi + \gamma a)\tau - (\beta + \gamma a)]Y. \tag{4}$$

Second, we differentiate the natural capital stock per worker E/L with respect to time

$$\dot{e} = \frac{d(E/L)}{dt} = \frac{\dot{E}L - E\dot{L}}{L^2} \Rightarrow \dot{e} = \frac{\dot{E}}{L} - e\frac{\dot{L}}{L}. \tag{5}$$

By replacing equation (4) in equation (5), we get the differential equation

$$\dot{e} = [(\phi + \gamma a)\tau - (\beta + \gamma a)]k^{\eta}e^{\theta} - [n(L) - \alpha]e.$$

Setting $A = (1 - a)(1 - \tau) > 0,\ B = (\phi + \gamma a)\tau - (\beta + \gamma a),$ we have that the model's economy is described by the following system of non-linear differential equations

$$\begin{cases} \dot{k} = Ak^{\eta}e^{\theta} - [\delta + n(L)]k, \\ \dot{e} = Bk^{\eta}e^{\theta} - [n(L) - \alpha]e, \\ \dot{L} = Ln(L). \end{cases} \tag{6}$$

Given $k_0 = k(0) > 0,\ e_0 = e(0) > 0,$ this Cauchy problem has a unique solution, denoted by $(k(t),\ e(t),\ L(t)),$ defined on $[0, \infty)$ (see Birkhoff and Rota [1]).

3. Long-run Sustainability Conditions

In the neoclassical Solow model, there is no mention of environmental resources, and the implicit assumption is that environmental stock is fixed, and it does not depend on human activities. An economy is said to be long run sustainable so long as per capita consumption c equals or exceeds a given subsistence consumption level $\bar{c} > 0,$ i.e., $k^{\eta}e^{\theta} \geq \bar{c}/[a(1 - \tau)],$ for all t. In particular, $k(t)$ and $e(t)$ must be both at least positive. Here, the natural capital is an essential input in the production process. Thus, the prosperity and survival of the economy will depend on its ability to manage the environment. It is clearly unwise for the economy to develop by running down the environment indefinitely. Considering the environment as a private goods, i.e., no joint consumption, we have that long run sustainability also requires that the environmental stock never falls below a minimum life-sustaining level $\bar{e} > 0,$ i.e., $e(t) \geq \bar{e},$ for all t. In general, there is no guarantee that the stock of physical capital as well as the stock of natural capital will remain positive as time grows indefinitely large. We are now going to show that this depends crucially on the sign of B.

Proposition 1. *Let* $B = 0$.

(i) *For all t, the natural capital is described by the function*

$$e(t) = e_0 \exp(\alpha t) L(t)^{-1}, \tag{7}$$

where exp denotes the exponential function. As t grows to infinity, the function $e(t)$ *decreases monotonically to* 0 *or to* $e_0 L_\infty^{-1}$ *if* $\alpha < 0$ *or* $\alpha = 0$, *respectively, while it diverges to* $+\infty$ *if* $\alpha > 0$.

(ii) *For all t, the physical capital is described by the function*

$$k(t) = [\exp(\delta t) L(t)]^{-1} \left[k_0^{1-\eta} + A e_0^{\theta} g(t)\right]^{\frac{1}{1-\eta}} \tag{8}$$

where $g(t) = \int_0^t \exp\{[\alpha\theta + (1-\eta)\delta] t\} L(t)^{1-(\eta+\theta)} dt$. *In the long run,* $k(t)$ *converges to* 0 *or to* $\left[A e_0^{\theta} / (1-\eta)\delta L_\infty^{\theta}\right]^{\frac{1}{1-\eta}}$ *if* $\alpha < 0$ *or* $\alpha = 0$, *respectively, while it diverges to* $+\infty$ *if* $\alpha > 0$.

Proof. (i) $B = 0$ in (6) yields the separable differential equation

$$\dot{e} = -[n(L) - \alpha] e. \tag{9}$$

It is immediate to check that $\dot{e} < 0$ if $\alpha \leq 0$, and that (7) is the unique solution of (9). Setting $\lim_{t \to \infty} e(t) = e_\infty$, then it is clear from (7) that $e_\infty = 0$ if $\alpha < 0$, $e_\infty = e_0 L_\infty^{-1}$ if $\alpha = 0$, and $e_\infty = +\infty$ if $\alpha > 0$.

(ii) Plugging (7) in $\dot{k} = A k^{\eta} e^{\theta} - [\delta + n(L)] k$ of (6) yields the Bernoulli differential equation $\dot{k} = A[e_0 \exp(\alpha t) L(t)^{-1}]^{\theta} k^{\eta} - [\delta + n(L)] k$. Taking the substitution $z = k^{1-\eta}$ yields a linear differential equation in z, whose solution expressed in terms of k is provided by (8). In order to understand the long run behavior of the function $k(t)$, let us rewrite (8) as

$$k(t)^{1-\eta} = \frac{k_0^{1-\eta} + A e_0^{\theta} g(t)}{\exp[(1-\eta)\delta t] L(t)^{1-\eta}}. \tag{10}$$

Let $\alpha \geq 0$. As t grows to infinity, the right hand side of (10) leads to an indeterminate form since both its numerator and denominator go to infinity. This fact is immediate for the denominator since $1 - \eta > 0$ and $1 \leq L_\infty < \infty$, while for the numerator it follows from the inequality

$$g(t) \geq \int_0^t \exp\{[\alpha\theta + (1-\eta)\delta]t\}dt = \frac{\exp\{[\alpha\theta + (1-\eta)\delta]t\} - 1}{\alpha\theta + (1-\eta)\delta},$$

and the fact that $\alpha\theta + (1-\eta)\delta > 0$. To solve this indeterminate form, we apply Hopital's rule. This gives

$$\lim_{t\to\infty} k(t)^{1-\eta} = \frac{Ae_0^\theta}{(1-\eta)\delta}\left[\lim_{t\to\infty}\frac{\exp(\alpha t)}{L(t)}\right]^\theta,$$

i.e.,

$$\lim_{t\to\infty} k(t) = \left\{\frac{Ae_0^\theta}{(1-\eta)\delta}\left[\lim_{t\to\infty}\frac{\exp(\alpha t)}{L(t)}\right]^\theta\right\}^{\frac{1}{1-\eta}}.$$

The statement of the Proposition is now immediate recalling that L_∞ is finite.

Proposition 2. *Let* $B < 0$. *Then the function* $e(t)$ *is monotone decreasing. As t grows to infinity,* $e(t)$ *converges to zero if* $\alpha < 0$, *it does not diverge to infinity if* $\alpha = 0$, *its behavior is unknown if* $\alpha > 0$; $k(t)$ *does not converge to zero no matter who is* α. *Let* $B > 0$. *Then nothing can be concluded about the long run behavior of the functions* $e(t)$ *and* $k(t)$.

Proof. *Let* $B < 0$. If $\alpha \leq 0$, then $\dot{e} = Bk^\eta e^\theta - [n(L) - \alpha]e < 0$, i.e., $e(t)$ is monotone decreasing. To study the long run behavior of the functions $e(t)$, $k(t)$, we should be able to solve the differential equations of (6) in terms of elementary functions, as done in the case $B = 0$. However, this is now not possible. In this case, a common technique is to compare the unknown solutions of the given equations with the known solutions of another, i.e., to use the so-called Comparison theorems (see Birkhoff and Rota [1]). Using the following theorem: "if $u_i(t)$, $i = 1, 2$, is the solution of the Cauchy problem $\dot{u} = \varphi_i(t, u), u(0) = u_0,$ and $\varphi_1(t, u) \leq \varphi_2(t, u),$ for all $(t, u),$ then $u_1(t) \leq u_2(t),$ for all t", we have the second part of the statement of our Proposition. For example, from Proposition 1 and the inequality $\varphi_1 = Bk^{\eta}e^{\theta} - [n(L) - \alpha]e < -[n(L) - \alpha]e = \varphi_2,$ we get that $e(t) < e_0 \exp(\alpha t) L(t)^{-1},$ for all t. Thus, $e_{\infty} = 0$ if $\alpha < 0,$ and $e_{\infty} < e_0 L_{\infty}^{-1}$ if $\alpha = 0.$ Similarly, in order to understand the long run behavior of the function $k(t),$ we need to study the differential equation $\dot{k} = Ak^{\eta}e^{\theta} - [\delta + n(L)]k.$ From $\varphi_1 = -[\delta + n(L)]k < Ak^{\eta}e^{\theta} - [\delta + n(L)]k = \varphi_2,$ it follows that $k_0[\exp(\delta t) L(t)]^{-1} < k(t),$ for all t. Thus, $\lim_{t \to \infty} k(t) > 0.$

We can now state the following result.

Theorem 1. *If human activities have a net zero or negative effect on the environment in every time period and the stock of the environment decays autonomously over time, i.e., if* $B \leq 0$ *and* $\alpha < 0,$ *then the economy is unsustainable in the long run. If human activities have a net zero effect on the environment and the stock of the environment grows or remains unchanged autonomously over time, i.e.,* $B = 0$ *and* $\alpha \geq 0,$ *then the economy is sustainable in the long run.*

Remark 2. In case of a constant population growth rate, i.e., $\dot{L}/L = n,$ and the hypothesis $n > \alpha,$ Tran-Nam [9] showed that the economy is always unsustainable in the long run if $B \leq 0.$ Moreover, a necessary condition for the economy to be long run sustainable is that $B > 0,$ i.e., if human activities produce a net beneficial effect on the environment for every time period.

4. Tax Rate and Sustainability

Theorem 1 implies that a sufficient condition for the economy to be sustainable in the long run is that $B = 0$ and $\alpha \geq 0,$ while a necessary condition is provided by $B < 0$ and $\alpha \geq 0,$ or $B > 0$ and α arbitrary. Mathematically, these conditions can be translated as follows.

Lemma 1.

(i) *For any given tax rate* $\tau \in (0, 1)$, $B \gtreqless 0$ *if and only if* $a \lesseqgtr (\phi\tau - \beta)/(1 - \tau)\gamma$.

(ii) *For any given MPC* $a \in (0, 1)$, $B \gtreqless 0$ *if and only if* $\tau \gtreqless (a\gamma + \beta)/(a\gamma + \phi)$.

(iii) $(\phi\tau - \beta)/(1 - \tau)\gamma \leq 0$ *if and only if* $\tau \leq \beta/\phi$;

(iv) $(\phi\tau - \beta)/(1 - \tau)\gamma \in (0, 1)$ *if and only if* $\beta/\phi < \tau < (\beta + \gamma)/(\phi + \gamma)$;

(v) $(\phi\tau - \beta)/(1 - \tau)\gamma \geq 1$ *if and only if* $\tau \geq (\beta + \gamma)/(\phi + \gamma)$.

Proof. The first part of the statement is immediate recalling that $B = (\phi + \gamma a)\tau - (\beta + \gamma a)$. The second part is an easy calculation.

Lemma 2. *If* $B \geq 0$, *then* $\phi > \beta$, *while if* $B < 0$ *the relationship between* ϕ *and* β *is undetermined.*

Proof. If $B = 0$, then $\phi\tau - \beta = (1 - \tau)a > 0$, i.e., $\phi\tau > \beta$. Since $0 < \tau < 1$, we have that $\phi\tau < \phi$. Thus, $\beta < \phi\tau < \phi$, i.e., $\phi > \beta$. If $B > 0$, then $\phi\tau - \beta > (1 - \tau)a > 0$. We now proceed as done before.

Proposition 3.

(1) *Let* $B > 0$ (α *arbitrary*). *For any given tax rate* $\tau \in (0, 1)$, *the set of sustainable MPCs is empty if* $\tau \leq \beta\phi$, *it is the interval* $(0, (\phi\tau - \beta)/(1 - \tau)\gamma)$ *if* $\beta/\phi < \tau < (\beta + \gamma)/(\phi + \gamma)$, *it is the interval* $(0, 1)$ *if* $\tau \geq (\beta + \gamma)/(\phi + \gamma)$.

(2) *Let* $B < 0$ $(\alpha \geq 0)$. *Let* $\phi > \beta$. *For any given tax rate* $\tau \in (0, 1)$, *the set of sustainable MPCs is* $(0, 1)$ *if* $\tau \leq \beta/\phi$, *it is* $((\phi\tau - \beta)/(1 - \tau)\gamma, 1)$ *if* $\beta/\phi < \tau < (\beta + \gamma)/(\phi + \gamma)$, *it is empty if* $\tau \geq (\beta + \gamma)/(\phi + \gamma)$. *Let* $\phi \leq \beta$. *For any given tax rate* $\tau \in (0, 1)$, *the set of sustainable MPCs is* $(0, 1)$.

(3) *For any given MPC* $\alpha \in (0, 1)$, *the set of sustainable tax rates is* $((\alpha\gamma + \beta)/(a\gamma + \phi), 1)$ *if* $B > 0$ (α *arbitrary*), *it is* $(0, (a\gamma + \beta)/(a\gamma + \phi))$ *if* $B < 0$ $(\alpha \geq 0)$ *and* $\phi > \beta$ *it is* $(0, 1)$ *if* $B < 0$ $(\alpha \geq 0)$ *and* $\phi \leq \beta$.

(4) *Let* $B = 0$ $(\alpha \geq 0)$. *For any given tax rate* τ, *the set of sustainable MPCs reduces to* $a = (\phi\tau - \beta)/(1 - \tau)\gamma$. *Similarly, for any given MPC* a, *the set of sustainable tax rates consists of only one element,* $\tau = (a\gamma + \beta)/(a\gamma + \phi)$.

Proof. (1) Let $\tau \leq \beta/\phi$. If there were a sustainable MPC $a \in (0, 1)$, then Lemma 1, (i), (iii) would imply that $a < (\phi\tau - \beta)/(1 - \tau)\gamma \leq 0$, i.e., $a < 0$. Thus, the set of sustainable MPCs is empty. Let $\beta/\phi < \tau < (\beta + \gamma)/(\phi + \gamma)$. Lemma 1 yields $a < (\phi\tau - \beta)/(1 - \tau)\gamma$, and $(\phi\tau - \beta)/(1 - \tau)\gamma \in (0, 1)$. The statement follows recalling that $a \in (0, 1)$. Finally, let $\tau \geq (\beta + \gamma)/(\phi + \gamma)$. Using again Lemma 1, we see that $a < (\phi\tau - \beta)/(1 - \tau)\gamma < 1 < (\phi\tau - \beta)/(1 - \tau)\gamma$. Consequently, the set of sustainable MPCs is $(0, 1)$. Similarly for the remaining cases.

Remark 3. Some interesting things can be derived from the previous Proposition. For example, if $B > 0$, we deduce that an increase in the tax rate in the relevant range widens the choice of sustainable MPCs, while a decrease in the tax rate narrows this choice. It is in fact clear that more resources are spent to repair the environment, then, keeping the economy sustainable, a larger fraction of the remaining output is available for consumption, while less resources are spent to repair the environment, then a smaller fraction of the remaining output is available for consumption.

5. Sustainable Steady State

A steady state of a sustainable economy is defined as a situation in which the growth rates of the per capita physical capital, the per capita natural capital, and the labor growth rate are equal to zero. Let us denote the steady state equilibrium values of k, e, L, by k_*, e_*, L_*, respectively. In studying the steady states of our sustainable economy, we will confine our analysis to interior steady states only, i.e., we will exclude the economically meaningless solutions such as $k_* = 0$, $e_* = 0$, or $L_* = 0$.

Proposition 4.

(1) *No steady states of* (6) *exist if* $B = 0$ $(\alpha > 0)$, *if* $B < 0$ $(\alpha = 0)$, *or if* $B > 0$ $(\alpha \geq 0)$.

(2) *There is a unique steady state of* (6) *if* $B < 0$ $(\alpha > 0)$, *or if* $B > 0$ $(\alpha < 0)$:

$$(k_*, e_*, L_*) = (\omega, -(\delta B/\alpha A)\omega, L_*),$$

where $\omega = [(A/\delta)^{1-\theta}(-B/\alpha)^{\theta}]^{1/[1-(\eta+\theta)]}$.

(3) *There are infinite steady states of* (6) *if* $B = 0$ $(\alpha = 0)$;

$$(k_*, e_*, L_*) = (k, [(\delta/A)k^{1-\eta}]^{1/\theta}, L_*), \textit{ for all } k > 0.$$

Proof. The steady states equilibrium will be determined by the conditions $\dot{k} = \dot{e} = \dot{L} = 0$. From (6), we have the following system of equations

$$Ak^{\eta-1}e^{\theta} = \delta, Bk^{\eta}e^{\theta-1} = -\alpha, L = L_*. \tag{11}$$

Let $B = 0$. Then (11) becomes $Ak^{\eta-1}e^{\theta} = \delta$, $\alpha = 0$, $L = L_*$. Hence, there cannot be steady states if $\alpha \neq 0$. Let $\alpha = 0$, Since $Ak^{\eta-1}e^{\theta} = \delta$, we get $e = [(\delta/A)k^{1-\eta}]^{1/\theta}$. Thus, there exists a steady state for each $k > 0$. Let $B < 0$. If $\alpha = 0$, then (11) implies that $Bk^{\eta}e^{\theta-1} = 0$, an absurd since the left hand side of this equality is a negative number. If $\alpha > 0$, then from (11) we get that $e = -(\delta B/\alpha A)k$, and $k = [(A/\delta)^{1-\theta}(-B/\alpha)^{\theta}]^{1/[1-(\eta+\theta)]}$. Consequently, there is a unique steady state. Similarly the proof for the case $B > 0$.

Remark 4. Let $B = 0$ $(\alpha = 0)$. We know from Proposition 1 that in the long run the economy converges to $(k_\infty, e_\infty, L_\infty)$, where $k_\infty = [Ae_0^{\theta}/(1-\eta)\delta L_\infty^{\theta}]^{\frac{1}{1-\eta}}$, $e_\infty = e_0 L_\infty^{-1}$. Since this point is not a steady state, we have that the economy stabilizes to a point different from a steady state equilibrium in the long run. If we suppose that there is a value $k > 0$ such that $(k_{\infty}, e_{\infty}, L_{\infty}) = (k, [(\delta/A)k^{1-\eta}]^{1/\theta}, L_*)$, then we would obtain $(1-\eta)\delta = \delta$, i.e., an absurd.

Theorem 2. *If* $B = 0$ $(\alpha = 0)$, *every steady state equilibrium is unstable. If* $B < 0$ $(\alpha > 0)$, *the unique steady state equilibrium is a saddle with a two dimensional stable manifold. If* $B > 0$ $(\alpha < 0)$, *the unique steady state equilibrium is a stable node.*

Proof. The local dynamic around (k_*, e_*, L_*) is determined by the signs of the eigenvalues of the Jacobian matrix corresponding to its linearized system, which writes

$$\begin{bmatrix} \dot{k} \\ \dot{e} \\ \dot{L} \end{bmatrix} = J^* \begin{bmatrix} k - k_* \\ e - e_* \\ L - L_* \end{bmatrix}, \text{ where } J^* = \begin{bmatrix} J_{11}^* & J_{12}^* & J_{13}^* \\ J_{21}^* & J_{22}^* & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{33}^* \end{bmatrix}.$$

J^* is the Jacobian matrix of the system (6) evaluated at (k_*, e_*, L_*). By definition, it is $J_{11}^* = (\partial\dot{k}/\partial k)|_{(k_*, e_*, L_*)}$, $J_{12}^* = (\partial\dot{k}/\partial e)|_{(k_*, e_*, L_*)}$, $J_{13}^* = (\partial\dot{k}/\partial L)|_{(k_*, e_*, L_*)}$, and so on for all the other matrix entries. Since $J_{11}^* = -(1-\eta)\delta$, $J_{12}^* = A\theta k_*^{\eta} e_*^{\theta-1}$, $J_{13}^* = -n_L(L_*)k_*$, $J_{21}^* = B\eta\delta/A$, $J_{22}^* = (1-\theta)\alpha$, $J_{23}^* = -n_L(L_*)e_*$, $J_{31}^* = J_{32}^* = 0$, and $J_{33}^* = n_L(L_*)L_*$, we have that

$$J^* = \begin{pmatrix} -(1-\eta)\delta & A\theta k_*^{\eta} e_*^{\theta-1} & -n_L(L_*)k_* \\ B\eta\delta/A & (1-\theta)\alpha & -n_L(L_*)e_* \\ 0 & 0 & n_L(L_*)L_* \end{pmatrix}.$$

It is immediate that one eigenvalue of this matrix, say λ_1, equals $n_L(L_*)L_*$. Note that λ_1 is real and negative. Let λ_2, λ_3 denote the remaining two eigenvalues of J^*. The signs of these two eigenvalues can be derived looking at the trace and the determinant of J^*, where the determinant of J^* is

$$\mathrm{Det}(J^*) = [-(1-\eta)(1-\theta)\delta\alpha - B\theta\eta\delta k_*^{\eta} e_*^{\theta-1}]n_L(L_*)L_*,$$

and the trace of J^* is given by

$$\mathrm{Trace}(J^*) = J_{11}^* + J_{22}^* + J_{33}^* = -(1-\eta)\delta + (1-\theta)\alpha + n_L(L_*)L_*.$$

Recalling that the determinant of a matrix is also equal to the product of its eigen-values, as well as the trace of a matrix is also equal to the sum of its eigenvalues, we obtain

$$\lambda_2\lambda_3 = -(1-\eta)(1-\theta)\delta\alpha - B\theta\eta\delta k_*^{\eta} e_*^{\theta-1},$$

$$\lambda_2 + \lambda_3 = -(1-\eta)\delta + (1-\theta)\alpha. \tag{12}$$

Note that λ_2, λ_3 must be real numbers. Let $B = 0(\alpha = 0)$. Then (12) becomes

$$\lambda_2\lambda_3 = 0, \lambda_2 + \lambda_3 = -(1-\eta)\delta < 0.$$

Therefore, one of the two eigenvalues must be negative, the other must be null. Let $B \neq 0$. Since (11) implies that $k_*^{\eta} e_*^{\theta-1} = -\alpha/B$, it follows that (12) rewrites as

$$\lambda_2\lambda_3 = -[1-(\theta+\eta)]\delta\alpha, \lambda_2 + \lambda_3 = -(1-\eta)\delta + (1-\theta)\alpha.$$

If $B < 0$ $(\alpha > 0)$, then $\lambda_2\lambda_3 < 0$, i.e., λ_2, λ_3 must have opposite sign. From $\lambda_2 + \lambda_3 < 0$, we derive that one eigenvalue is positive, the other is negative. We can now conclude that the system is saddle-path stable since its three eigenvalues have different signs. The stable manifold will be a plane going through the steady state since there are two negative eigenvalues. If $B > 0$ $(\alpha < 0)$, then $\lambda_2\lambda_3 > 0$, i.e., λ_2, λ_3 must have the same sign. Since $\lambda_2 + \lambda_3 < 0$, we deduce that both the eigenvalues are negative. Consequently, since all the three eigenvalues are negative, the unique steady state equilibrium is a stable node.

Remark 5. In case of a constant population growth rate, Tran-Nam [9] showed that, if human activities produce a net beneficial effect on the environment, then the economy will converge to a unique and stable steady state.

6. Conclusion

In this paper, we presented an effort to incorporate natural capital into the neoclassical Solow-Swan model with no technological innovation under the assumption of a variable population growth rate. The natural capital stock is modeled as a renewable resource. In this framework, we find out that the economy is sustainable in the long run if human activities have a net zero effect on the environment and the stock of the environment grows or remains unchanged autonomously over time, while the economy is unsustainable if human activities have a net zero or negative effect on the environment and the stock of the environment decays autonomously over time. For any given tax rate (or MPC, i.e., marginal propensity to consume), we derive the set of sustainable MPCs (or tax rates). Finally, we examine the non-trivial steady states of a sustainable economy, and discover that there are infinite unstable steady states equilibrium if human activities have a net zero effect on the environment and the environment remains unchanged over time, while there is a unique stable steady state equilibrium, which is a saddle, or a node, if human activities have a negative effect on the environment and the environment grows over time, or if human activities have a positive effect on the environment and the environment decays over time, respectively.

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